Section 7.1

given diagonal matrix $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ determine:

rank $A = 3$

determinant A = product of diagonal entries $= 0$

basis of ker A

$$\text{rref} A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ 0 \\ 0 \end{bmatrix} = r \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_2}$$

augmented

let $x_2 = $ arb. r

$\therefore$ basis of ker A = span $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

find $A^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix}$

$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$

def'n: A is <u>diagonalizable</u> iff $A \sim B$ s.t. B is a diagonal matrix
i.e. $\exists$ invertible S s.t. $S^{-1}AS = B$

revisit: given $\vec{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$ $S = \begin{array}{c} \begin{bmatrix} 2 & 5 \\ 5 & 12 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 \end{matrix} \end{array}$, determine $[\vec{x}]_{\mathcal{B}}$

$\vec{x} = S[\vec{x}]_{\mathcal{B}}$

$\det S = 24 - 25$
$= -1$

$S^{-1} = -\begin{bmatrix} 12 & -5 \\ -5 & 2 \end{bmatrix}$

$S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$

$-\begin{bmatrix} 12 & -5 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 16 \end{bmatrix} = -\begin{bmatrix} 84-80 \\ -35+32 \end{bmatrix} = -\begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} = [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

<u>check</u> $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$

$-4\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 12 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 7 \\ 16 \end{bmatrix}$

$\begin{bmatrix} -8+15 \\ -20+36 \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$ ✓

revisit: given $A = \begin{bmatrix} 1 & 2 \\ \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ \end{bmatrix}$

$\begin{bmatrix} \\ -20+36 \end{bmatrix} = \begin{bmatrix} \\ 16 \end{bmatrix} \vec{v}$

revisit: given $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

find B of $T(\vec{x}) = A\vec{x}$ w.r.t. $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$

$S = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ $S^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

if $A \sim B$ then

$AS = SB \rightarrow B = S^{-1}AS$

$= \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ do matrix multiplication

$B = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ or $7\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

given $T(\vec{x}) = A\vec{x} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$

new scalar notation:

lambda

B is diagonal iff $A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$ for scalars $\lambda_1, \dots, \lambda_n$

$$B = \begin{array}{c} \begin{array}{cccc} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{array} \\ \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \end{array} \begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{array}$$

generalize

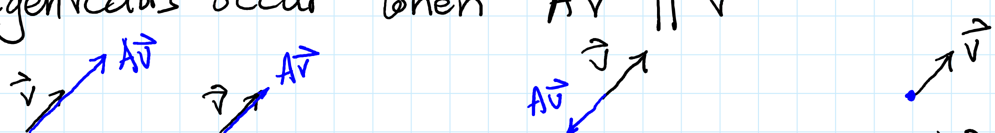
$$\boxed{A\vec{v} = \lambda\vec{v}}$$

non-zero vector, $\vec{v} \in \mathbb{R}^n$ is an eigenvector of A if $A\vec{v} = \lambda\vec{v}$

and scalar, λ, is an eigenvalue of $\vec{v}$

basis $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ is an eigenbasis for A if $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors

i.e. eigenvectors occur when $A\vec{v} \parallel \vec{v}$



$A\vec{v} = 2\vec{v}$ eigenvalue: 2

$A\vec{v} = 1\vec{v}$ eigenvalue: 1

$A\vec{v} = -\vec{v}$ eigenvalue: -1

$A\vec{v} = \vec{0} = 0\vec{v}$ eigenvalue: 0

as was seen previously: given $A\vec{v} = \lambda\vec{v}$

then $A^2\vec{v} = \lambda^2\vec{v}$

$\vdots$

$A^m\vec{v} = \lambda^m\vec{v}$

$$A^m \vec{v} = \lambda^m \vec{v}$$

A is diagonalizable iff $\exists$ eigenbasis for A

if $\vec{v}_1, \dots, \vec{v}_n$ is an eigenbasis for A with $A\vec{v}_1 = \lambda_1 \vec{v}_1, \dots, A\vec{v}_n = \lambda_n \vec{v}_n$

then $S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ and $B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$ will diagonalize A s.t. $S^{-1}AS = B$

ex. What are the possible real eigenvalues of orthogonal $n \times n$ matrix A?

A is orthogonal $\Rightarrow T(\vec{x}) = A\vec{x}$ preserves length

$$\|T(\vec{x})\| = \|A\vec{x}\|$$

$$\|\vec{x}\| \quad \forall x \in \mathbb{R}^n$$

eigenvalue

given $A\vec{v} = \lambda \vec{v}$ ← eigenvector

then $\|\vec{v}\| = \|A\vec{v}\| = \|\lambda \vec{v}\|$

$= |\lambda| \, \|\vec{v}\|$

$|\lambda| = 1 \Rightarrow \lambda = \pm 1$

ex. given reflection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A\vec{e}_1 = \vec{e}_1$ $\quad$ $A\vec{e}_2 = -\vec{e}_2$

$\lambda_1 = 1$ $\quad$ $\lambda_2 = -1$

thm: orthogonal matrix can only possibly have real eigenvalue of -1 or 1

Section 7.2

use $A\vec{v} = \lambda \vec{v}$

$A\vec{v} - \lambda \vec{v} = 0$

$A\vec{v} - \lambda I_n \vec{v} = 0$

$(A - \lambda I_n)\vec{v} = 0$

then $\ker(A - \lambda I_n) \neq \{\vec{0}\}$ i.e. $\exists \vec{v}$ in kernel besides zero vector

thm: λ is an eigenvalue of $n \times n$ matrix A iff $\det(A - \lambda I_n) = 0$

thm: λ is an eigenvalue of $n \times n$ matrix A iff $\det(A - \lambda I_n) = 0$

characteristic equation

ex. find eigenvalue(s) of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

use $\det(A - \lambda I_2) = 0$

$$\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \qquad \lambda I_2 = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0 \quad \leftarrow \text{ad} - \text{bc}$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$3 - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda - 5)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = 5 \quad \lambda_2 = -1}$$

ex. find eigenvalues of $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

$$\det(A - \lambda I_3) = 0$$

$$\det\left(\begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{bmatrix} = 0 \quad \leftarrow \text{upper tri. matrix}$$

$$(2-\lambda)(3-\lambda)(4-\lambda) = 0$$

$$\boxed{\lambda_1 = 2 \quad \lambda_2 = 3 \quad \lambda_3 = 4}$$

generalize:

thm: eigenvalues of a triangular matrix are its diagonal entries

ex. find characteristic equation for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A-\lambda I_2) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad \underbrace{- a\lambda - d\lambda} + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

when $\lambda = 0$: $\quad ad - bc = \det(A - \lambda I_n)$

coefficient of λ-term : $-(a+d)$, which is additive inverse of sum of diagonal entries a, d of A

def'n trace := sum of diagonal entries of square matrix A

notation: $tr(A)$

then characteristic equation of 2×2 matrix A:

$$\det(A-\lambda I_2) = \lambda^2 - (tr A)\lambda + \det A = 0$$

revisit when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then $tr A = 1+3 = 4$

$\det A = 3-8 = -5$

then char. eqn : $\lambda^2 - 4\lambda - 5 = 0$

$(\lambda-5)(\lambda+1) = 0$

$\lambda_1 = 5 \quad \lambda_2 = -1$

highest power of polynomial

given $n\times n$ matrix A, $\det(A-\lambda I_n)$ is a polynomial w/ degree $= n$:

$$(-\lambda)^n + (trA)\lambda^{n-1} + \dots + \det A$$

$$= (-1)^n \lambda^n + (-1)^{n-1}(trA)\lambda^{n-1} + \dots + \det A \quad \leftarrow \text{called characteristic polynomial}$$

notation: $f_A(\lambda)$